Approximate Solution of Second-Order Nonlinear Differential Equations

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Nonlinear differential equations of the form:

$$\ddot{x} + h(x,\dot{x})\dot{x} + k^2(x,\dot{x})x = 0 \qquad (1)$$

encountered in many problems of mechanics and radio engineering cannot be integrated in known functions and must be solved by approximate methods of averaging.

In the most wide-spread variant of this method, proposed by van der Pol [1], we set

$$x = a.\sin(wt + e) \qquad (2)$$

[Note: 'w' represents 'omega', the circular frequency, later a function of t: $\omega(t)$.]

where $\underline{t}$ is the time appearing as the independent variable in equation (1), $\underline{w}$ is a certain constant to be determined, a and e are slowly varying functions of $\underline{t}$ which are connected by the relation:

$$\dot{a}\sin(wt + e) + a\dot{e}\cos(wt + e) = 0 \qquad (3)$$

For $\underline{\dot{a}}$ and $\underline{\dot{e}}$ one obtains the following "abbreviated" equations averaged over the period $2\pi/w = T$

$$\dot{a} = -(a/2\pi)\int_0^{2\pi} h(a\sin u,\ aw\cos u)\cos^2 u\,du$$
$$\dot{e} = (1/2\pi w)\int_0^{2\pi} k^2(a\sin u,\ aw\cos u)\sin^2 u\,du - w/2 \qquad (4)$$

Hence e and a are found by quadrature.

Let us dwell for a moment on the partial case where the coefficients $\underline{k^2}$ and $\underline{h}$ depend only upon $\underline{x}$. Then the first equation (4) assuming the form

$$\dot{a} = -(a/2\pi)\int_0^{2\pi} h(a\sin u)\cos^2 u\,du \qquad (5)$$

gives the dependence of the instantaneous amplitude a upon time, and the second equation in (4) represented in the form

$$w + \dot{e} = w/2 + (1/2\pi w) \int_0^{2\pi} k^2(a \sin u) \sin^2 u \, du \tag{6}$$

determines the instantaneous frequency $w + \dot{e}$.

If one can so select the constant w that in the course of the entire process considered $\dot{e}$ remains of small megnitude in comparison with w, then we represent equation (6) in the form

$$w^2 + 2w\dot{e} = (1/\pi)\int_0^{2\pi} k^2(a \sin u) \sin^2 u \, du \tag{7}$$

and add to the left side of it a small quantity $\dot{e}^2$ of the second order, thus obtaining the formula for the square of the instantaneous frequency

$$(w + \dot{e})^2 = (1/\pi)\int_0^{2\pi} k^2(a \sin u) \sin^2 u \, du \tag{8}$$

the right side of which does not depend the selection of the constant w.

If the frequency of the process under consideration varies considerably in the course of time, then $\dot{e}$ becomes a quantity of the same order of magnitude as w and the substitution of formula (6) by formula (8) loses its foundation. Use of the formula (6) on the one hand leads to a physically absurd conclusion — namely, the instantaneous frequency $w + \dot{e}$ becomes depend upon the selection of the constant w and always remains greater than w/2.

Such results are clarified by the fact that if $\dot{e}$ becomes small,* then e cannot be considered a slowly varying function of time; consequently one of the main premises of the method under consideration collapses and the method becomes useless.

What has been said remains true even for the refined variant of the method of averaging given by Bulgakov[2].

*[Note: Sic.]

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For processes described by equations of the type (1), but differing by the fact that their frequency in slowly varying deviates over a sufficiently long period of time considerably from the initial frequency, it is convenient to take the following new variation of the method of averaging. We set

$$x = a \cdot \sin \left(\int_0^t w\,dt + e \right) \qquad (9)$$

where a,w,e are slowly varying functions of time; here w will also be considered a slowly varying function of time. Affixing to these functions the condition:

$$\dot{a} \cdot \sin \left(\int_0^t w\,dt + e \right) + a\dot{e} \cdot \cos \left(\int_0^t w\,dt + e \right) = 0 \qquad (10)$$

and substituting the expression (9) into equation (1) we obtain

$$a \left\{ k^2 \left[a \cdot \sin T(t),\ aw \cdot \cos T(t) \right] - w^2 - w\dot{e} \right\} \cdot \sin T(t) + \left\{ a\dot{w} + \dot{a}w + awh \left[a \sin T(t),\ aw \cos T(t) \right] \right\} \cos T(t) = 0 \qquad (11)$$

where $T(t) = \int_0^t w\,dt + e$.

Determining a and e from equations (10) and (11) and averaging the formulas obtained over the period of the trigonometric functions entering these functions, we are led to the euqations

$$\dot{a} = -(a\dot{w}/2w) - (a/2\pi)\int_0^{2\pi} h(a \sin u,\ aw \cos u) \cos^2 u\,du \qquad (12)$$

$$\dot{e} = (1/2\pi w)\int_0^{2\pi} k^2 (a \cdot \sin u,\ aw \cdot \cos u) \sin^2 u\,du - w/2$$

We require now that w should be averaged over the period of circular frequency of the process and consequently that in the average over the period we should have

$$\dot{e} = 0. \qquad (13)$$

Then from the second equation (12) we obtain

$$w^2 = \frac{1}{\pi}\int_0^{2\pi} k^2 (a \sin u,\ aw \cdot \cos u) \cdot \sin^2 u\,du \qquad (14)$$

The first equation (12) after the substitution

$$\dot{w} \equiv \dot{a} dw/da \tag{15}$$

takes the final form

$$\dot{a}(1 - adw/2wda) = -(a/2\pi)\int_0^{2\pi} h(a\cdot\sin u, aw\cdot\cos u)\cdot\cos^2 u du \tag{16}$$

After this $\underline{w}$ is determined from equation (14) in dependence upon $\underline{a}$ and is substituted in equation (16); the latter becomes a differential equation with the variables $\underline{a}$ and $\underline{t}$ separated out and integrated by quadrature. Here one constant of integration enters the solution, and the other constant of intergration is $\underline{e}$.

As an example let us consider the differential equations of a system of regulation with a nonlinear servomotor

$$\dot{\varphi} = \xi, \quad \sigma = -(\varphi + \xi), \quad T\dot{\xi} = \sigma\sqrt{|\sigma|}\,. \tag{17}$$

Eliminating the variables φ and ξ, we obtain the differential equation for σ

$$\ddot{\sigma} + \frac{3}{2T}\sqrt{|\sigma|}\cdot\dot{\sigma} + \frac{1}{T}\sqrt{|\sigma|}\cdot\sigma = 0\,. \tag{18}$$

Formula (14) gives for this case

$$\omega^2 = \frac{4}{4T}\sqrt{\alpha}\int_0^{\pi/2}\sin^{\frac{5}{2}} u\cdot du = \frac{2\Gamma(\frac{7}{4})\Gamma(\frac{1}{2})}{\pi\Gamma(\frac{9}{4})}\cdot\frac{\sqrt{\alpha}}{T} = \frac{0.916}{T}\sqrt{\alpha}\,. \tag{19}$$

Equation (16) assumes the form: $\dot{\alpha}\left(1+\frac{1}{8}\right) = -3\alpha^{3/2}/\pi T\int_0^{\pi/2}\sin^{\frac{1}{2}} u\cos^2 u du = -\frac{3\Gamma(\frac{3}{4})\Gamma(\frac{3}{2})}{2\pi\,\Gamma(\frac{9}{4})}\cdot\frac{\alpha^{3/2}}{T} = -\frac{0.458}{T}\alpha^{3/2}$. Hence $\frac{1}{\sqrt{\alpha}} = \frac{4\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{3\pi\Gamma(\frac{9}{4})}\cdot\frac{t+C}{T} = \frac{0.204}{T}(t+C)$* (21)

Substitution of the expression for α into formula (19) gives:

$$\omega = \frac{3}{\sqrt{2(t+C)}}\,. \tag{22}$$

Finally we obtain according to formula (9):

$$\sigma = \frac{24.2T^2}{(t+C)^2}\cdot\sin\left[3\sqrt{2(t+C)} + e\right]. \tag{23}$$

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*Note: C is a constant of integration.

The variable sigma executes damped oscillations whose frequency decreases as the oscillations die away and tends toward zero.

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2. B. V. Bulgakov. Kolebaniya (Oscillations), GTTI. 1949. Vol I

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